Physics for Bangladesh: Quantum Field Theory Lecture Notes

Nabil Iqbal

September 10, 2024

Contents

1 Orientation

These are lecture notes written for the Physics for Bangladesh Quantum Field Theory online school, which took place in Fall 2024; lecture videos for that course are available here.

They are based on a (very) abridged version of a course previously taught at Durham University on path integral methods for quantum field theory, together with some extra material on path integrals for QM. The full lecture notes for that course can be found here.

We will use the "mostly-plus" metric:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1) \tag{1.1}$$

This is the opposite to the convention I am used to, so everyone has to remain vigilant for sign errors. We will mostly work in four spacetime dimensions (the "physical" value), but it is nice to keep the spacetime dimension d arbitrary where possible.

Note that Greek indices μ, ν will run over time and space both, whereas i, j will run only over the three spatial coordinates, and thus

$$x^\mu = (x^0, x^i) \tag{1.2}$$

I will sometimes use x^0 to denote the time component and sometimes x^t, depending on what I feel looks better in that particular formula.

I will always set $\hbar = c = 1$.[1] They can be restored if required from dimensional analysis.

2 Path integrals for quantum mechanics

Up till now you have studied quantum field theory by writing down fields as operators $\hat{\phi}$ etc. and then imposing commutation relation on them. This approach – the "usual" one – for understanding a quantum system is called **canonical quantization**. As it turns out, there is an alternative approach, involving what is called a **path integral**: this is a very beautiful formulation of the physics, and like most things in physics that are beautiful it also turns out to be quite practically useful in calculating things.

However before studying path integrals in quantum field theory, we will begin by studying path integrals for quantum mechanics. My discussion here is taken mostly from Peskin and Schroeder Chapter 9 and Anthony Zee Chapter 1[2].

2.1 Many holes in many screens

I want to motivate this by thinking about the double slit experiment, which everyone knows. To remind you, imagine a slit with two holes A_1 and A_2, with a source S and an observer O. What is the probability to find a particle after it has gone through the slit? By the superposition principle, we know that the amplitude to find the particle at the end is

$$\mathcal{A}(O) = \mathcal{A}(S \to A_1 \to O) + \mathcal{A}(S \to A_2 \to O) \tag{2.1}$$

i.e. the sum of the amplitudes to go through either of the slits. (In elementary quantum mechanics we now usually expand out these amplitudes explicitly in terms of plane waves, but we will not do that here).

[1] k_B makes an appearance in a homework problem (or rather it would, if I hadn't set it to 1).

[2] This book is amazing if you want to be motivated to do anything.

Now imagine that we add another hole A_3. Then, well the amplitude is clearly:

$$\mathcal{A}'(O) = \mathcal{A}(S \to A_1 \to O) + \mathcal{A}(S \to A_2 \to O) + \mathcal{O}(S \to A_3 \to O) \tag{2.2}$$

But now what if we add A_4 and A_5? Clearly we should add those two. But what if we add another screen, with more slits B_i? (Sum up those paths as well). What if we fill *all of space* with screens? What if we put so many holes in the screen that the screen is no longer there?

What all of this is really suggesting is that in general, should be a way to compute the amplitude in terms of **summing up all the ways that the particle can go from S to O.** Apparently each of them will contribute with some probability, and we should this should seem philosophically soothing: what is quantum mechanics, after all, but the idea that a particle always does everything at once, with a given probability?

This philosophical musing can be elevated to physics we have a formula for it: in other words, how much does each path contribute, and how do we "sum over all paths" anyway?

2.2 A simple finite dimensional integral to warm up

As it turns out, we will sum over paths by doing many integrals. Let's warm up first. There are very few integrals that we can do in closed form. One of them – a crucial one – is the Gaussian integral in one variable.

$$\int_{-\infty}^{\infty} dx e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}} \tag{2.3}$$

Also, if we add a linear shift in x in this form, the integral is still easy to do:

$$\int_{-\infty}^{\infty} dx e^{-\frac{ax^2}{2} - jx} = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{j^2}{2a}\right) \tag{2.4}$$

There is a basic generalization of this integral. I promise we will need it later, but I'm going to record it for you now. Imagine that $x_i \in \mathbb{R}^N$ and consider the following integral:

$$I = \int \prod_{i=1}^{N} dx^i \exp\left(-\frac{x_i A_{ij} x_j}{2}\right) \tag{2.5}$$

where A is an $N \times N$ matrix. After doing a small amount of work (homework problem!) you can show that this integral is

$$I = \int \prod_{i=1}^{N} dx^i \exp\left(-\frac{x_i A_{ij} x_j}{2}\right) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}} \tag{2.6}$$

Hint: diagonalize the matrix A and work in terms of its eigenvalues. Extra hint: this is worked out in Zee's book Quantum Field Theory in a Nutshell. Finally, we will need a generalization of this – let me consider adding an extra vector $J_i \in \mathbb{R}^N$ to this, so that we have

$$I[J] = \int \prod_{i=1}^{N} dx^i \exp\left(-\frac{x_i A_{ij} x_j}{2} + J_i x_i\right) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}} \exp\left(+\frac{1}{2} J_i (A^{-1})_{ij} J_j\right), \tag{2.7}$$

where the last equality can be found by completing the square.

2.3 Derivation of path integral for quantum mechanics

We now present an actual derivation in the case of quantum mechanics. At the end the generaliztion to quantum field theory will actually be very simple.

Consider a quantum system with one coordinate $\hat{q}$, canonical momentum $\hat{p}$, and Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{2.8}$$

where $V(\hat{q})$ is some potential which we don't need to fix yet.

We would like to derive a formula for the following object: the probability for a particle to go from q_a to q_b in time T, i.e.

$$\langle q_b|e^{-i\hat{H}T}|q_a\rangle \tag{2.9}$$

where $|q_{a,b}\rangle$ are position eigenstates, i.e. they satisfy the equation $\hat{q}|q_a\rangle = q_a|q_a\rangle$. We will also need momentum eigenstates later, which satisfy the eauation $\hat{p}|p_a\rangle = p_a|p_a\rangle$.

To derive a formula for this, the first thing we will do is introduce a very small time interval ϵ and break the whole time T up into $N \equiv T/\epsilon$ small chunks, so that we write:

$$e^{-i\hat{H}T} = e^{-i\hat{H}\epsilon}e^{-i\hat{H}\epsilon}\cdots e^{-i\hat{H}\epsilon} \tag{2.10}$$

The next amazingly clever thing that we will do is, in between *each* of the small ϵ pieces, we will add a complete set of states, i.e.

$$\mathbf{1} = \int dq_k|q_k\rangle\langle q_k| \tag{2.11}$$

where k runs from 1 to $(N-1)$. In other words this now looks like

$$\langle q_b|e^{-i\hat{H}T}|q_a\rangle = \langle q_b|e^{-i\hat{H}\epsilon}\int dq_{N-1}|q_{N-1}\rangle\langle q_{N-1}|e^{-i\hat{H}\epsilon}\int dq_{N-2}|q_{N-2}\rangle\langle q_{N-2}|e^{-i\hat{H}\epsilon}\int dq_{N-3}|q_{N-3}\rangle\langle q_{N-3}|\cdots\langle q_1|e^{-i\hat{H}\epsilon}|q_a\rangle \tag{2.12}$$

We can move all the integrals over to the left – thus we see that the thing to do is to evaluate many matrix elements of the form

$$\langle q_{k+1}|e^{-i\hat{H}\epsilon}|q_k\rangle \tag{2.13}$$

In this notation its natural to rename $q_b = q_N$ and $q_a = q_0$.

The answer depends on the exact form of H. Let's start by doing it for the simple case of a free particle which is just

$$H = \frac{\hat{p}^2}{2m} \tag{2.14}$$

We want to calculate

$$\langle q_{k+1}|e^{-i\frac{\hat{p}^2}{2m}\epsilon}|q_k\rangle \tag{2.15}$$

What is the matrix of the momentum operator in between two position eigenstates? To understand this, let's first recall that

$$\langle q|p\rangle = e^{iqp} \qquad \int \frac{dp}{2\pi}|p\rangle\langle p| = \mathbf{1} \tag{2.16}$$

The first equality just reminds us that the momentum eigenstate in the position basis is a plane wave. Now we introduce a complete set of states to find

$$\langle q_{k+1}|e^{-i\frac{\hat{p}^2}{2m}\epsilon}|q_k\rangle = \int \frac{dp}{2\pi}\langle q_{k+1}|e^{-i\frac{\hat{p}^2}{2m}\epsilon}|p\rangle\langle p||q_k\rangle \tag{2.17}$$

Now, we can act with the momentum operator on the eigenstate to just get that $\frac{\hat{p}^2}{2m}|p\rangle = \frac{p^2}{2m}|p\rangle$ (note the loss of the hat!) and then take this number outside. Similarly, we use the inner product between $\hat{q}$ and $\hat{p}$ to find

$$\langle q_{k+1}|e^{-i\frac{\hat{p}^2}{2m}\epsilon}|q_k\rangle = \int \frac{dp}{2\pi}e^{-i\frac{p^2\epsilon}{2m}}e^{ip(q_{k+1}-q_k)} \tag{2.18}$$

We are left with a Gaussian integral over p. But we know how to do Gaussian integrals, so we just refer back to (2.4) find

$$\langle q_{k+1}|e^{-i\frac{\hat{p}^2}{2m}\epsilon}|q_k\rangle = \left(\frac{-im}{2\pi\epsilon}\right)^{\frac{1}{2}} \exp\left(\frac{im(q_{k+1}-q_k)^2}{2\epsilon}\right) = \left(\frac{-im}{2\pi\epsilon}\right)^{\frac{1}{2}} \exp\left(\frac{im\epsilon}{2}\left(\frac{q_{k+1}-q_k}{\epsilon}\right)^2\right) \tag{2.19}$$

Now recall that this was just one of the many little *bits* that made up the full integral. Assembling all of them together we find

$$\langle q_b|e^{-i\hat{H}T}|q_a\rangle = \left(\frac{-im}{2\pi\epsilon}\right)^{\frac{N}{2}} \int \prod_{k=1}^{N-1} dq_k \exp\left(\frac{im\epsilon}{2}\sum_{k=1}^{N-1}\left(\frac{q_{k+1}-q_k}{\epsilon}\right)^2\right) \tag{2.20}$$

Now, magically, we take the limit $\epsilon \to 0$. Recall that ϵ was a small increment in time. Note that from the definition of the derivative and Riemann sum of integrals

$$\lim_{\epsilon\to 0}\sum \epsilon \to \int dt \qquad \lim_{\epsilon\to 0}\frac{q_{k+1}-q_k}{\epsilon} \to \frac{dq}{dt} \tag{2.21}$$

and thus we can write the truly beautiful expression

$$\boxed{\langle q_b|e^{-i\hat{H}T}|q_a\rangle = \int [\mathcal{D}q] \exp\left(i\int_0^T dt\frac{\dot{q}^2}{2}\right)} \tag{2.22}$$

where the "integral over paths" is now formally defined as

$$\int [\mathcal{D}q] = \lim_{N\to\infty}\left(\frac{-im}{2\pi\epsilon}\right)^{\frac{N}{2}} \int \prod_{k=1}^{N-1} dq_k \qquad q(0) = q_a \qquad q(T) = q_b \tag{2.23}$$

This is an amazing formula: let's think about it for a second. What it says is that to compute the propagation amplitude $\langle q_b|e^{-i\hat{H}T}|q_a\rangle$, you should integrate over *all possible paths* that the particle can take from q_a to q_b, where we also have a precise definition of the integral over paths in terms of a discretization (2.23). There is also a precise formula for how much each path contributes – it contributes with a phase factor $e^{i\int_0^T dt\frac{\dot{q}^2}{2}}$.

Now we did this calculation for a free Hamiltonian with $\hat{H} = \frac{\hat{p}^2}{2m}$; however we can now imagine putting the potential $V(q)$ back. It is not too hard to show that if we have the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(q) \tag{2.24}$$

then the calculation above is modified to read

$$\langle q_b|e^{-i\hat{H}T}|q_a\rangle = \int [\mathcal{D}q] \exp\left(i\int_0^T dt\left(\frac{\dot{q}^2}{2} - V(q)\right)\right) \tag{2.25}$$

Note the crucial sign difference in front of the $V(q)$. We normally write this as

$$\langle q_b|e^{-i\hat{H}T}|q_a\rangle = \int [\mathcal{D}q] \exp\left(i\int_0^T dt L(q,\dot{q})\right) \qquad L(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - V(q) \tag{2.26}$$

i.e. it is the integral of the Lagrangian – i.e. the *action* S – which appears in the exponent!

This should again feel soothing. After all, classical physics can be studied in either the Lagrangian or the Hamiltonian framework. But when we do quantum mechanics we usually only use the Hamiltonian. You may have wondered what the "Lagrangian" way of doing quantum mechanics is. Now you know.

Now sometimes we want to compute not just the propagation amplitude but also perhaps some expectation values – e.g. how do we calculate this?

$$\langle q_b|e^{-iH(T-t_2)}\hat{q}e^{-iH(t_2-t_1)}\hat{q}e^{-iHt_1}|q_a\rangle \qquad T > t_2 > t_1 > 0 \tag{2.27}$$

If you now go through the *exact* same derivation again, you will see that the insertion of the operators means that when you do the path integral you should also evaluate the coordinate being path-integrated over at some specific points, i.e.

$$\langle q_b|e^{-iH(T-t_2)}\hat{q}e^{-iH(t_2-t_1)}\hat{q}e^{-iHt_1}|q_a\rangle = \int [\mathcal{D}q]q(t_2)q(t_1)\exp\left(i\int_0^T dtL(q,\dot{q})\right) \tag{2.28}$$

There is something important happening here about the operator ordering on the left-hand side – note that on the right hand side $q(t_2)$ and $q(t_1)$ are just numbers (specific parts of a path integral that we are integrating over) – so it doesn't matter what order you put them in. But on the left hand side they were quantum operators where the ordering clearly matters.

How do we figure out what order to put them in? If you think about it, the ordering was actually determined by time – t_2 is to the left of t_1 because it is later in time (i.e. closer to the final endpoint T). In other words, expectation values that we compute from a path integral always give us **time-ordered** correlation functions, and sometimes we express this by saying

$$\langle q_b|T(\hat{q}(t)\hat{q}(t'))|q_a\rangle = \int [\mathcal{D}q]q(t)q(t')\exp\left(i\int_0^T dtL(q,\dot{q})\right) \tag{2.29}$$

where we are now writing the operators $\hat{q}$ in the Heisenberg picture.

Finally, note that this tells us how to go from q_a to q_b. However, we will often be interested not in going from q_a to q_b but rather in going from one state $|I\rangle$ to another state $|F\rangle$. It's easy to write down a general formula for that (exercise!). Let's ask about something even simpler though – what if we want to go from the vacuum $|0\rangle$ back to the vacuum $|0\rangle$? One way to do this is to note first that every state can be decomposed in energy eigenstates $|n\rangle$ with energies E_n, including the position eigenstate $|q_a\rangle$:

$$|q_a\rangle = \sum_n c_n|n\rangle \tag{2.30}$$

Let us set the zero of energy so that $E_0 = 0$ (i.e. the ground state has energy zero). Now note that if we act on this with $e^{-i\hat{H}T(1-i\epsilon)}$, with ϵ a small infinitesimal that is unrelated to that of the previous section, then we find:

$$e^{-i\hat{H}T(1-i\epsilon)}|q_a\rangle = \sum_n c_n e^{-iE_nT-\epsilon TE_n}|n\rangle \tag{2.31}$$

Finally, consider taking $T\to\infty$ – in that case the exponential damping by $e^{-\epsilon TE_n}$ kills all of the excited states with $n > 0$, leaving behind only the ground state!

$$\lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}|q_a\rangle \to c_0|0\rangle \tag{2.32}$$

The point of this is just that its very easy to get the vacuum – just evolve in *slightly* imaginary time! So this gives us a simple way to evaluate the following object:

$$\langle 0|0\rangle \propto \lim_{T\to\infty}\langle q_b|e^{-i\hat{H}T(1-i\epsilon)}|q_a\rangle = \lim_{T\to\infty(1-i\epsilon)}\int [\mathcal{D}q]\exp\left(i\int_0^T dtL(q,\dot{q})\right) \tag{2.33}$$

At this point it might not be clear why we are doing this – the thing to take away from here is that **if we make time slightly imaginary, then we project onto the vacuum**. This will be useful later. So, for example, if we want to calculate

$$\boxed{\langle 0|T(\hat{q}(t_1)\hat{q}(t_2))0\rangle = \frac{\lim_{T\to\infty(1-i\epsilon)} \int [\mathcal{D}q] q(t_1) q(t_2) \exp\left(i\int_0^T dt L(q,\dot{q})\right)}{\lim_{T\to\infty(1-i\epsilon)} \int [\mathcal{D}q] \exp\left(i\int_0^T dt L(q,\dot{q})\right)}} \tag{2.34}$$

Here the division by the thing without the q's inserted is important because usually we don't actually manage to keep track of the right normalization of the state.

2.4 Mathematical interlude

Above we have explained how to *integrate* over functions of one variable $\int[\mathcal{D}q]$ – basically we did it by discretizing it into a series of ordinary integrals and then doing each of them. This is sometimes called *functional integration* (though I will usually call it a *path integral*).

There is another thing that we will often need to do, which is take a *derivative* with respect to a function. Let's explain how this works, first from normal calculus. Imagine that we have a vector $x^i \in \mathbb{R}^N$. Then the derivative obviously works like this:

$$\frac{\partial x^i}{\partial x^j} = \delta_{ij} \tag{2.35}$$

This leads to interesting expressions like this:

$$\frac{\partial}{\partial x^j} e^{\sum_i A^i x^i} = \frac{\partial}{\partial x^j}\left(\sum_k A^k x^k\right) e^{\sum_i A^i x^i} \tag{2.36}$$

$$= \left(\sum_k A^k \delta_{kj}\right) e^{\sum_i A^i x^i} \tag{2.37}$$

$$= A_j e^{\sum_i A^i x^i} \tag{2.38}$$

So, note what happened – I took a derivative with respect to one of the variables (x_j), and this brought down the variable multipling it (A_j). Here I am writing out the sums instead of using the Einstein summation convention for a reason which will be evident shortly.

Now – we now need to move this up slightly by considering how we take derivatives with respect to *functions*. A function $q(t)$ is basically just many many variables. (If this bothers you, imagine discretizing the coordinate t, as we did in the first lecture for space x). So we can imagine a **functional derivative** which acts like this

$$\frac{\delta q(t)}{\delta q(t')} = \delta(t-t') \tag{2.39}$$

This is just the fancy functional version of (2.35). For physicists this expression completely defines the functional derivative and you can use it to calculate anything you need. Note that in this analogy $q(t)$ is just like x^i and $q(t')$ is just like x^j, and the delta function δ_{ij} is like $\delta(t-t')$.

Continuing in this way, note that

$$\frac{\delta}{\delta j(t)} \int dt' j(t') q(t') = \int dt' \delta(t-t') q(t') = q(t) \tag{2.40}$$

And finally, note from the same reasoning that

$$\frac{\delta}{\delta j(t)} \exp\left(\int dt' j(t') q(t')\right) = q(t) \exp\left(\int dt' j(t') q(t')\right) \tag{2.41}$$

We will use this in what follows.

3 Path integrals in free quantum field theory

With this under our belt, we will now move on to quantum field theory. Now that we understand the quantum mechanics case this is really extremely simple. We will begin with a study of the free real scalar field ϕ. This has action

$$S[\phi] = \frac{1}{2}\int d^4x \left((\partial\phi(x))^2 - m^2\phi(x)^2\right) = \frac{1}{2}\int d^4x\ \phi(x)(-\partial^2 - m^2)\phi(x) \tag{3.1}$$

The classical equations of motion arising from the variation of this action are

$$(\partial^2 + m^2)\phi(x) = 0 \tag{3.2}$$

This is philosophically the same as the quantum mechanics problem studied earlier. To make the transition imagine going from the single quantum-mechanical variable q to a large vector q_a where a runs (say) from 1 to N. Now imagine formally that a runs over all the sites of a lattice that is a discretization of space, and now $q_a(t)$ is basically the same thing as $\phi(x^i;t)$ which is exactly the system we are studying above.

Now we would like to study the quantum theory. First, I point out that we can define a path integral in precisely the same way as before, i.e. we can consider the following path-integral:

$$Z_0 \equiv \int [\mathcal{D}\phi] \exp\left(iS[\phi]\right) \tag{3.3}$$

Where $S[\phi]$ is the action written down above, and $[\mathcal{D}\phi]$ now represents the functional integral over all *fields* and not just particle trajectories.

There are two main things that are nice about doing quantum field theory from path integrals the way discussed above. One of them is honest, the other is a bit "secret".

1. The honest one: all of the symmetries of the problem are manifest. The action S is Lorentz-invariant, and it is fairly easy to see how these symmetries manifest themselves in a particular computation. Compare this to the Hamiltonian methods used in the first half of the courst, where you have to pick a time-slice and it always seems like a miracle when final answers are Lorentz-invariant.

2. The secret one: the path integral allows one to be quite cavalier about subtle issues like "what is the structure of the Hilbert space exactly". This is very convenient when we get to gauge fields, where there are subtle constraints in the Hilbert space (google "Dirac bracket") that you can more or less not worry about when using the path integral (i.e. one can go quite far in life without knowing exactly what a "Dirac bracket" is).

3.1 The generating functional

Now that the philosophy is out of the way, let us do a computation. We will begin by computing the following two-point function:

$$\langle 0|T(\phi(x)\phi(y))|0\rangle \tag{3.4}$$

This object is called the *Feynman propagator*. It is quite important for many reasons; I will discuss them later, for now let's just calculate it. By arguments identical to those leading to (2.34), we see that we want

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = Z_0^{-1}\int [\mathcal{D}\phi]\phi(x)\phi(y) \exp\left(iS[\phi]\right) \tag{3.5}$$

To calculate this, it is convenient to define the same generating functional as we used for quantum mechanics

$$Z[J] \equiv \int [\mathcal{D}\phi] \exp\left(iS[\phi] + i\int d^4x J(x)\phi(x)\right) \tag{3.6}$$

And we then see from functional differentiation that the two-point function is

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right) Z[J]\Big|_{J=0} \tag{3.7}$$

Each functional derivative brings down a $\phi(x)$. Now we will evaluate this function. We first note the following crucial identity from a few hours ago, which I have embellished with a few i's here and there

$$\int \prod_{i=1}^{N} dx^i \exp\left(-\frac{x_i A_{ij} x_j}{2} + iJ_i x_i\right) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}} \exp\left(-\frac{1}{2} J_i (A^{-1})_{ij} J_j\right) \tag{3.8}$$

We note from the form of the action (3.1) that the path integral $Z[J]$ we want to do is of precisely this form, where we do our usual "many integrals" limit and where a labels points in space and the operator A is

$$A = i\left(\partial^2 + m^2\right) \tag{3.9}$$

We conclude that the answer for $Z[J]$ is

$$Z[J] = \det\left(\frac{\partial^2 + m^2}{-2\pi i}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\int d^4x d^4y J(x) D_F(x,y) J(y)\right) \tag{3.10}$$

where I have given the object playing the role of A^{-1} a prescient name D_F. It is the inverse of the differential operator defined in (3.9) and thus satisfies

$$i\left(\partial^2 + m^2\right) D_F(x,y) = \delta^{(4)}(x-y) \tag{3.11}$$

This is an important result. Let us first note that the path integral is asking us to compute the *functional determinant* of a differential operator. This is a product over infinitely many eigenvalues; it is quite a beautiful thing but we will not really need it here, so we will return to it later.

The next thing to note is that the dependence on J is quite simple; the exponential of a quadratic. Indeed, inserting this into (3.7) we get

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = D_F(x,y) \tag{3.12}$$

Thus, we have *derived* that the time-ordered correlation function of $\phi(x)$ is given by the inverse of $i(\partial^2 + m^2)$!

Let us now actually calculate this object. We first go to Fourier space:

$$D_F(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \tilde{D}_F(p) \tag{3.13}$$

Inserting this into (3.11) we find

$$\int \frac{d^4p}{(2\pi)^4}\left(-p^2 + m^2\right) e^{-ip\cdot(x-y)} \tilde{D}_F(p) = -i\delta^{(4)}(x-y) \tag{3.14}$$

We now see that we want the object in momentum space $\tilde{D}_F(p)$ to be proportional to $p^2 - m^2$ so that we can use the identity $\int d^4p e^{ip\cdot x} = (2\pi)^4 \delta^{(4)}(x)$. Getting the factors right, we find the following expression for the propagator in Fourier space:

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2} \tag{3.15}$$

This is the propagator that we know and love from the earlier part of the course.

Now we can use path integrals to re-derive the Feynman rules – this is a different way of doing it then we did it using the interaction picture. Sadly there is no time to re-explain it here.

End of first lecture.

4 Abelian Gauge Theories

Now that we have understood fermions, we will move on to gauge theories. We warm up with Abelian gauge theories before moving on to the full glory and beauty of non-Abelian gauge theories for the final part of the course.

4.1 Gauge invariance

We will now first understand what a gauge symmetry is, starting from the simplest example. First consider a theory with a global $U(1)$ symmetry: it is nice to consider the Dirac theory to start,

$$S[\psi, \bar{\psi}] = \int d^4x \; \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) \tag{4.1}$$

This theory is clearly invariant under a global $U(1)$ phase rotation, which is

$$\psi(x) \to \psi'(x) = e^{i\Lambda}\psi(x) \qquad \bar{\psi}(x) \to e^{-i\Lambda}\bar{\psi}(x) \tag{4.2}$$

I emphasize that the symmetry parameter Λ here is a constant in spacetime, which is why we call it a *global* symmetry. I also want to emphasize that two field configurations that are *related* by the symmetry are both *physical field configurations*. If ψ represents something that we integrate over in the path integral, then so does ψ'.

Now here is an idea, which I simply state without a huge amount of motivation: let's do something different. Let us instead *demand* that we have a theory that is invariant under a symmetry where the symmetry parameter *varies* in spacetime:

$$\boxed{\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)} \tag{4.3}$$

This is called a *local* symmetry, or a *gauge symmetry*. As I will explain later, it should probably really be called a *gauge redundancy*, but I will not use that language in this course and continue to call it a gauge symmetry. Things that are invariant under this symmetry are called *gauge-invariant*.

Now, let's understand what is and is not gauge-invariant. For example: $\psi(x)$ itself is *not* gauge invariant, but it transforms under gauge transformations in a simple way (by an overall rotation).

The mass term

$$m\bar{\psi}'(x)\psi'(x) = m\bar{\psi}(x)e^{-i\Lambda(x)}e^{+i\Lambda(x)}\psi(x) \to m\bar{\psi}(x)\psi(x) \tag{4.4}$$

clearly *is* gauge invariant. Note that it would not have been if the field and its conjugate were evaluated at different points, as then the gauge parameter would not have canceled.

Now how about the derivative term in the action? Now we run into an issue that has to do with the local character of the gauge transformation: note that if we try to take a derivative we find

$$\partial_\mu\psi'(x) = \partial_\mu\left(e^{i\Lambda(x)}\psi(x)\right) = e^{i\Lambda(x)}\left(i\partial_\mu\Lambda + \partial_\mu\right)\psi(x) \tag{4.5}$$

Something very bad has happened because $\Lambda(x)$ depends on space, we have picked up an extra term $\partial_\mu\Lambda$. There is no obvious way to get rid of this, and thus we conclude that the Dirac action as written (4.12) is *not* gauge invariant.

This is a similar problem: the derivative of ψ no longer has nice gauge transformation properties. To fix it,we will do the same thing that we did previously: we define a new object, called the *gauge covariant derivative*:

$$\boxed{D_\mu\psi \equiv \left(\partial_\mu + ieA_\mu\right)\psi} \tag{4.6}$$

where I have introduced a new field A_μ, called the *gauge field* or the *gauge connection*. (GR students may be tempted to compare it to the Christoffel connection: this analogy will get even closer when we do non-Abelian gauge theories soon). I have also chosen to take out a factor e for later convenience.

We will *demand* that the gauge-covariant derivative of ψ has a nice gauge transformation property, i.e. for invariance purposes let us demand:

$$D_\mu\psi \to D'_\mu\psi' = e^{i\Lambda(x)}D_\mu\psi, \tag{4.7}$$

without any awkward inhomogenous piece. Let's see what we require A'_μ to do to make this happen. Expanding out we have

$$\begin{aligned} D'_\mu\psi' = \left(\partial_\mu + ieA'_\mu\right)e^{i\Lambda(x)}\psi(x) &= e^{i\Lambda(x)}\left(i\partial_\mu\Lambda + ieA'_\mu + \partial_\mu\right)\psi(x) & (4.8)\\ &= e^{i\Lambda(x)}\left(\partial_\mu + ieA_\mu\right)\psi(x) & (4.9)\end{aligned}$$

where the last equality is our demand. We see that this will work out if and only if:

$$\boxed{A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\Lambda(x) \qquad \psi'(x) = e^{i\Lambda(x)}\psi(x)}, \tag{4.10}$$

where I have also written down the transformation property of ψ, as the two are related. This is the *gauge-transformation of A_μ*: if it transforms in this way, then our gauge-covariant derivative does indeed do what we want it to do. Note that now the following term:

$$\bar{\psi}(x)\gamma^\mu D_\mu\psi(x) \to \bar{\psi}'(x)\gamma^\mu D'_\mu\psi'(x) = \bar{\psi}(x)e^{-i\Lambda(x)}\gamma^\mu e^{+i\Lambda(x)}D_\mu\psi(x) = \bar{\psi}(x)\gamma^\mu D_\mu\psi(x) \tag{4.11}$$

is gauge-invariant, and thus we have succeeded in taking a derivative in a gauge-invariant way. With this, we see that we can write a fully gauge-invariantized-Dirac-action:

$$S[\psi,\bar{\psi},A]_{\text{Dirac}} = \int d^4x\ \bar{\psi}(x)(i\not{D} - m)\psi(x) \tag{4.12}$$

where we just promote the partial to a gauge-covariant derivative. To do this, we had to invent another field called A_μ. This is the price of gauge-invariance.

But now that we have this field A_μ, we may ask what more we can do with it. To see this, let's first note that $D_\mu\psi$ has a nice gauge-transformation property, and thus so will $D_\mu D_\nu\psi$. It thus makes sense to consider the *commutator* of the two derivatives:

$$\begin{aligned} [D_\mu, D_\nu]\psi &= [\partial_\mu,\partial_\nu]\psi + ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi & (4.13)\\ &= ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\psi & (4.14)\end{aligned}$$

Notice that this commutator of derivatives is (when acting on ψ) actually not a derivative at all: it is just a number multiplying ψ. Let's call this number $ieF_{\mu\nu}$, i.e.

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{4.15}$$

$F_{\mu\nu}$ is called the *field-strength* of the gauge field. Now, purely abstractly, we know that $F_{\mu\nu}$ must be gauge-invariant. That's because we have

$$[D_\mu, D_\nu] = ieF_{\mu\nu} \tag{4.16}$$

Now consider acting this equation on ψ and then doing a gauge-transformation on both sides: we know how both sides transform

$$e^{i\Lambda(x)}[D_\mu, D_\nu]\psi = ieF_{\mu\nu}e^{i\Lambda(x)}\psi \tag{4.17}$$

and we can now cancel the $e^{i\Lambda(x)}$, which immediately implies that $F_{\mu\nu}$ is gauge invariant. Of course we could also simply directly use the gauge transformation property of A_μ (4.10) directly, but this more abstract viewpoint will be useful when we get to non-Abelian gauge theories.

It is not hard to convince yourself that random derivatives of A_μ, i.e $\partial_\mu A_\nu$ by itself, are actually not gauge-invariant. So let us now try to write down the most general renormalizable gauge-invariant action involving our new friend A_μ and $\psi(x)$. It turns out the most general parity-invariant[3] action is

$$S[\psi,\bar{\psi}] = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(x)(i\not{D} - m)\psi(x)\right) \tag{4.18}$$

The first term is gauge-invariant since $F_{\mu\nu}$ is gauge invariant. We have discussed the second term extensively.

How about a mass term for the gauge field, e.g.

$$M^2 A_\mu A^\mu \tag{4.19}$$

It is easy to see that this is not gauge-invariant, and thus one says that gauge-invariance forbids bare mass terms for gauge fields. Thus we conclude that the most general gauge-invariant and P-invariant action with a gauge field and a single fermion is just

$$\boxed{S[\psi,\bar{\psi},A]_{\text{QED}} = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(x)(i\not{D} - m)\psi(x)\right)} \tag{4.20}$$

This is the action of QED. Note that we have been led to it purely from symmetry principles: *there is nothing else that we could have written down.*

4.2 Some classical aspects of Abelian gauge theory

For a little while, let us focus on the theory given by just the first part of the action above. As you already know, this turns out to be ordinary Maxwell electrodynamics:

$$S[A]_{\text{Maxwell}} = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{4.21}$$

Let's figure out the classical equations of motion: varying this action with respect to A_μ we find that these equations of motion are just

$$\partial_\mu F^{\mu\nu} = 0 \tag{4.22}$$

If we pick a time direction and write this down in Lorentz-non-invariant notation $F^{0i} = E^i$, $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ then these are simply Maxwell's equations, and thus this is the Lagrangian for free EM with no matter.

Now, we should note an interesting fact: recall the Klein-Gordon equation of motion for a free scalar field

$$\partial^2\phi = 0 \tag{4.23}$$

This defines what is called a *well-defined Cauchy problem*, which means that if you pick a time $t = 0$ and specify initial data $\phi(t = 0, \vec{x})$ as well as $\partial_t\phi(t = 0, \vec{x})$, then you can use the equations of motion to propagate ϕ forwards in time unambiguiously. This is nice.

The fundamental degree of freedom for the Maxwell equation is $A_\mu(x)$, so this is the analog of $\phi(x)$. The above nice time evolution properties of the Klein-Gordon equation are *not* satisfied by the Maxwell equation. This can be understood from staring at the components of the equation, but there is an indirect route: note that if A_μ is a solution to the equations of motion, then so is $A_\mu - \frac{1}{e}\partial_\mu\Lambda(x)$, where Λ is any arbitrary function

[3] Ok, this is not strictly true, there is another parity-invariant action: you could imagine adding a term proportional to $\theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. P flips $\theta \to -\theta$; however it turns out that when θ is correctly normalized it is a periodic variable with periodicity 2π, which means that there is are *two* P-invariant points, one with $\theta = 0$ and one with with $\theta = \pi$, as $-\pi + 2\pi = \pi$. This seems like a ridiculous technicality that nobody should ever worry about, except that actually a lot of beautiful physics associated with something called a "topological insulator" arises from the $\theta = \pi$ point.

of spacetime. If we pick $\Lambda(t,\vec{x})$ to be a function that vanishes near $t=0$ but is different at late t, it is clear that the same initial data at $t=0$ can lead to two completely different A_μ's at late time, where they differ by a gauge transformation.

What do we do with this? It looks like our theory has no predictive power, in that knowing what happens at $t=0$ does not fix what happens at late t. We can save our classical theory by making the following assertion:

Things that are not gauge-invariant are *not physical.*

So it is okay that we cannot unambiguously solve for their time evolution. It is only things that are gauge-invariant – in this case, the components of $F_{\mu\nu}$ – that are physical, and you know from your elementary EM course that there is absolutely no issue solving for their time evolution.

Let's be a bit more explicit. Writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we find

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0 \tag{4.24}$$

As mentioned earlier, this is not a well-defined problem for time evolution of A_μ. We need to *fix a gauge*: one choice is Lorenz gauge[4]:

$$\partial_\mu A^\mu = 0 \tag{4.25}$$

(You can show that any A_μ can be placed in this form by a gauge transformation). If we do this, then we find the equation

$$\partial_\mu \partial^\mu A_\nu = 0, \tag{4.26}$$

which is now a perfectly nice wave equation that propagates all components of A_μ in time. It is also a wave equation with no mass term: in Fourier space, we get an equation that looks like:

$$(\omega^2 - k^2)A_\nu = 0 \tag{4.27}$$

and thus the on-shell energy of a single photon is $\omega_k = k$: the photon is massless.

However, it is still not true that all components are physical, as there is a *residual gauge invariance* in Lorenz gauge. I will not review here how this works, but you recall from elementary EM that for a plane wave with momentum k_μ it is the *transverse* components of A_μ to the momentum that are physical, and thus there are two physical polarizations. (One way to check this is to note that only those contribute to the field strength tensor; more formally you can find a way to fix the residual gauge invariance in Lorenz gauge).

With this classical understanding under control, we now turn to the quantum theory, where all these subtleties turn out to play a crucial role.

4.3 Quantizing QED

We turn now to the quantum theory. The new thing to do is to *quantize* the Maxwell action, i.e. we want to perform the following path integral:

$$Z = \int [\mathcal{D}A] \exp\left(i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right)\right) \tag{4.28}$$

We will ultimately be interested in quantizing the theory with the fermions also included, but first we will focus on this. As we understand in detail by now, the propagator is always equal to the inverse of the operator

[4] Poor Lorenz is an entirely different person from Loren*tz*.

appearing in the quadratic part of the Lagrangian: let us thus write this in a useful way:

$$S[A] = \int d^4x \left(-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\right) \tag{4.29}$$

$$= \int d^4x \frac{1}{2}\left(A_\nu \partial_\mu \partial^\mu A_\nu - A_\nu \partial^\nu \partial_\mu A^\mu\right) \tag{4.30}$$

$$= \int d^4x \frac{1}{2}\left(A_\mu \left(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu\right) \tag{4.31}$$

Now whenever we do a functional integral, we want to invert the differential operator that appears in the quadratic part of the action. Thus we seek to find a photon Feynman propagator $D^F_{\mu\nu}$ that satisfies:

$$\left(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu\right) D^F_{\nu\rho}(x,y) = i\delta^\mu_\rho \delta^{(4)}(x-y) \tag{4.32}$$

There is just one problem with this: no such $D^F_{\nu\rho}$ exists, because the differential operator *has no inverse*. This is perhaps clear if we go to momentum space, when this equation becomes

$$(p^2 \eta^{\mu\nu} - p^\mu p^\nu) D^F_{\mu\nu}(p) = -i\delta^\mu_\nu \tag{4.33}$$

But the matrix on the left-hand side has many zero eigenvalues: in fact, consider acting with it on any test function of the form $p_\mu \alpha(p)$:

$$(p^2 \eta^{\mu\nu} - p^\mu p^\nu) p_\mu \alpha(p) = \left(p^2 p^\nu - p^2 p^\nu\right) \alpha(p) = 0 \tag{4.34}$$

Thus the operator has a non-trivial kernel and cannot be inverted.

Physically, it should be clear that the issue is arising from the gauge transformations. The point is that any field configuration that is *pure gauge*

$$A_\mu = \frac{1}{e}\partial_\mu \Lambda \tag{4.35}$$

has field strength $F_{\mu\nu} = 0$, and thus has vanishing action. Thus when performing the path integral over functions in this "pure gauge" direction, the action does not oscillate and nothing suppresses their contribution to the path integral, which thus ends up diverging badly. This divergence is manifesting itself in the non-invertibility of the differential operator.

This is a serious issue. There is a correct way to fix this, called the *Fadeev-Popov procedure*, but sadly I have no time to really explain it – the end result is that we simply end up adding a term of this form to the action:

$$S_{\text{gauge-fixed}}[A] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right) \tag{4.36}$$

Thus the kinetic term has been modified, and of course now we can now invert the propagator. We go to Fourier space, and the equation to solve is then:

$$\left(-p^2 \eta_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu\right) D_F^{\nu\rho}(p) = i\delta^\rho_\mu \tag{4.37}$$

This sort of computation comes up again and again in field theory, so in this case I will work it out: consider the most general form of D_F. It is a tensor, and the only two tensor structures are $\eta^{\mu\nu}$ and $p^\mu p^\nu$, so we the answer must be of the form

$$D_F^{\nu\rho}(p) = A(p)\eta^{\nu\rho} + B(p)\frac{p^\nu p^\rho}{p^2} \tag{4.38}$$

where I have taken out some factors of p to make the final thing look nicer. Now we plug this ansatz into the equation and expand. We find

$$-p^2 A(p)\delta^\rho_\mu + p_\mu p^\rho \left(A(p)\left(1 - \frac{1}{\xi}\right) - \frac{B(p)}{\xi}\right) = i\delta^\rho_\mu \tag{4.39}$$

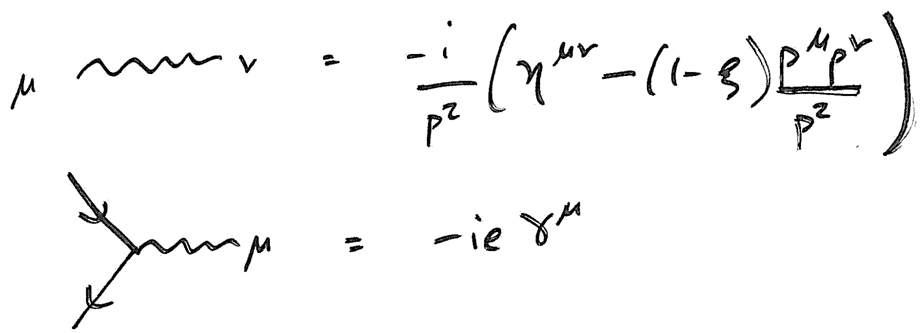


Figure 1: New Feynman rules involving photons

(Note taking away the gauge-fixing term corresponds to $\xi \to \infty$, in which case there is clearly no solution). We find that $A = -\frac{i}{p^2}$, $B = -\frac{i}{p^2}(\xi - 1)$, and thus:

$$\boxed{D_F^{\nu\rho}(p) = -\frac{i}{p^2}\left(\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)} \tag{4.40}$$

This is the Feynman propagator for photons, carefully derived.

What is ξ? The truth is: it is a gauge-fixing parameter, and so it *does not matter*. You can set it to any value you like, and you have to get the same answer for any calculation. This turns out to be guaranteed by the Ward identity in QED – basically dotting k^μ into any amplitude has to annihilate it. Thus people with truly formidable calculational powers will keep it arbitrary, and its cancellation at the end is a good check on the computation.

People (like me) with less formidable calculational powers will often find it convenient to set it to 1 (this is called Feynman gauge) as this makes the photon propagator have fewer terms in it.

This is it! Now you have carefully derived the photon propagator. This is the only tricky thing: from here we can rederive the full machinery of the Feynman rules. If we don't have the fermion then the theory is free; if we do have the fermion then the action is

$$S[\psi, \bar{\psi}, A]_{\text{QED}} = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(x)(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi(x)\right) \tag{4.41}$$

and thus we have the fermion Feynman rules that we wrote down last time and the following new rules for the photon:

- The photon propagator is $-\frac{i}{p^2}\left(\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$
- The photon-electron vertex is $-ie\gamma^\mu$.

(If you like, you can carefully derive the interaction vertex by doing the usual business with the sources, etc – no new effects enter from the gauge field).

It is important to note that the interaction vertex *came from the covariant derivative*, and thus we were not free to fix its form. This is an underlying principle: gauge symmetry fixes the possible interactions that can happen.

5 Non-Abelian gauge theories

Now that we understand Abelian gauge theories, we are going to move on to *non-Abelian* gauge theories. They are a powerful and nontrivial generalization of the same basic idea, and play a very important role in the Standard Model.

5.1 What is a group?

A tiny bit of abstraction first. What is a *group*? A group G is something that has elements in it $g_1, g_2, \cdots \in G$ so that there is some way to combine two elements to get a third element in the group. (There are also some formal properties – every element has an inverse, the identity element exists, etc. but these will be common sense for us). This combination is called the "group operation" and usually for us the operation will just be the *product*, i.e.

$$g_3 = g_1 g_2 \qquad g_{1,2,3} \in G \tag{5.1}$$

Here are some examples:

1. The group $U(1)$ generated by numbers $e^{i\alpha}$ with $\alpha \in \mathbb{R}$. Clearly if we have two of these and we take

   $$e^{i\alpha_1} e^{i\alpha_2} = e^{i\alpha_1 + i\alpha_2} \in U(1) \tag{5.2}$$

   Note that here it does not matter in which order we take the product,

   $$e^{i\alpha_2} e^{i\alpha_1} = e^{i\alpha_2 + i\alpha_1} = e^{i\alpha_1} e^{i\alpha_2} \tag{5.3}$$

   i.e. we say the group operation is **commutative** and we call this kind of group **Abelian**.

2. The group $SU(2)$, which means "unitary 2×2 matrices M that have determinant 1. Recall that unitary means $U^\dagger U = \mathbf{1}$. If we take two such $U_{1,2} \in SU(2)$ and take their matrix product, then we do get a third:

   $$U_1 U_2 = U_3 \in SU(2) \tag{5.4}$$

   However there is a very important distinction here: the group operation is now *not* Abelian, i.e. generically we have

   $$U_1 U_2 \neq U_2 U_1 \tag{5.5}$$

   Groups with this property – where the group action is *not commutative* – are called **non-Abelian**.

   It turns out everyone is intimately familiar with non-Abelian groups – e.g. this group $SU(2) \approx SO(3)$, i.e. the group of 2×2 unitary matrices is more or less (or rather, locally[5]) the same as the group of rotations of ordinary objects in 3d. And if you play with doing rotations in different orders, its quite clear that this group is not commutative.

It turns out that the previous part was constructing a gauge theory for the group $U(1)$. It turns out that we can make **a gauge theory for any group**. We will now construct a gauge theory for a non-Abelian group, where we will take the example of $SU(2)$.

5.2 Non-Abelian gauge invariance

An abstract way of formulating the discussion above was the following: a gauge transformation in the Abelian case was a map from spacetime to the group $U(1)$, i.e. something like $e^{i\Lambda(x)}$.

[5]The precise relation is that $SO(3) = SU(2)/\mathbb{Z}_2$, and this $\mathbb{Z}_2$ is basically the reason we have fermions.

Now we will generalize the whole thing to non-Abelian gauge theory. We will now try to formulate a theory that is invariant under non-Abelian gauge transformations, which are maps $U(x)$ from spacetime to the group $G = SU(2)$.

We will first need to understand how to discuss an arbitrary element of $SU(2)$. Though I say $SU(2)$ it turns out that everything will generalize to $SU(N)$ immediately with no changes. First, it is a fact that we can write an arbitrary element of $SU(2)$ like this:

$$U = \exp(i\alpha^a t^a) \tag{5.6}$$

where there are three t's, and they are the Pauli matrices $t^a = \frac{1}{2}\sigma^a$, or more explicitly:

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.7}$$

Exercise: check that this matrix is both unitary and has determinant 1.

These t's satisfy the following commutation relation:

$$[t^a, t^b] = i f^{abc} t^c \qquad f^{abc} = \epsilon^{abc} \tag{5.8}$$

(I am calling it f^{abc} and not ϵ^{abc} because this notation is standard in the non-Abelian gauge theory literature).

Now let us imagine having a fermion ψ that transforms like this under $SU(2)$ i.e. if $U(x) \in SU(2)$, we want to have the symmetry property:

$$\boxed{\psi(x) \to \psi'(x) = U(x)\psi(x),} \tag{5.9}$$

where ψ is now an 2-component complex vector, as well as a Dirac spinor (so it technically has 4×2 complex components, but we will suppress that information here).

Again, we can imagine writing down the Dirac Lagrangian:

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{5.10}$$

It is clear that this is invariant under $\psi(x) \to U\psi(x)$, $\bar{\psi}(x) \to \bar{\psi}(x)U^\dagger$ if U is constant; however, if $U(x)$ depends on space, then clearly the derivatives will not work out. The solution is clear: we now need to construct a non-Abelian gauge-covariant derivative.

We will do this as follows: the property that we want for the non-Abelian covariant derivative is

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = U(x) D_\mu \psi(x) \tag{5.11}$$

in other words, we want the whole thing to transform *homogenously*, i.e. with an overall factor of U from the left. From the $U(1)$ case, it should be clear that we will need a new object to do this: our new object is going to be the *non-Abelian gauge field*:

$$A_\mu^a(x) \tag{5.12}$$

where a runs from 1 to 3. The covariant derivative acting on ψ is taken to be

$$\boxed{D_\mu \psi(x) \equiv \left(\partial_\mu - igA_\mu^a t^a\right)\psi(x)} \tag{5.13}$$

where g is a number called the *gauge-coupling*, and is analogous to the e that we defined in the $U(1)$ case. Now let us see what property we require A_μ^a to have for (5.11) to work out:

$$D'_\mu \psi'(x) = \left(\partial_\mu - igA_\mu^{\prime a} t^a\right) U(x)\psi(x) \tag{5.14}$$

$$= (\partial_\mu U)\psi + U(x)\partial_\mu \psi - igA_\mu^{\prime a} t^a U(x)\psi(x) \tag{5.15}$$

$$\equiv U(x)\left(\partial_\mu - igA_\mu^a t^a\right)\psi(x), \tag{5.16}$$

where the last equality is what we *want* to be true. There is an interesting wrinkle here, because on the last line $U(x)$ is all the way on the left, whereas on the previous line it is always on the *right*, and because of the non-Abelian-ness we cannot commute it through. Nevertheless, we can still solve for $A_\mu^{'a}$ to find

$$\partial_\mu U - igA_\mu^{'a}t^aU = U(-igA_\mu^a t^a) \qquad \to \qquad \boxed{A_\mu^{'a}(x)t^a = U(x)\left(A_\mu^a(x)t^a + \frac{i}{g}\partial_\mu\right)U^\dagger(x)} \tag{5.17}$$

This is the transformation of the non-Abelian gauge field under a finite gauge transformation parametrized by $U(x)$. Before discussing what it means, it is instructive to also work out the *infinitesimal* version of the gauge transformation. To do this we parametrize $U(x)$ as

$$U(x) = \exp(i\alpha^a(x)t^a) \tag{5.18}$$

and then work to first order in α^a, i.e. we discard all terms of $\mathcal{O}(\alpha^2)$ or higher. The fermion transformation law (5.9) is simply

$$\psi'(x) = (1 + i\alpha^a(x)t^a)\,\psi(x) \tag{5.19}$$

whereas the gauge field one is slightly more intricate:

$$A_\mu^{'a}(x)t^a = \exp\left(i\alpha^b(x)t^b\right)\left(A_\mu^a(x)t^a + \frac{i}{g}\partial_\mu\right)\exp\left(-i\alpha^c(x)t^c\right) \tag{5.20}$$

$$= A_\mu^a(x)t^a + i[\alpha^b(x)t^b, A_\mu^a t^a] + \frac{1}{g}\partial_\mu\alpha^c t^c \tag{5.21}$$

$$= A_\mu^a(x)t^a - \alpha^b f^{bac}t^c A_\mu^a + \frac{1}{g}\partial_\mu\alpha^c t^c \tag{5.22}$$

Relabeling some indices and equating terms we find that the individual components of the gauge field transform as transform as

$$\boxed{A_\mu^{'a}(x) = A_\mu^a + f^{abc}A_\mu^b\alpha^c + \frac{1}{g}\partial_\mu\alpha^a} \tag{5.23}$$

So, what is going on? Here are a few important points:

1. The last term looks like the transformation of an Abelian gauge field.
2. The first term is different: what it means is that the non-Abelian gauge field points in a direction in group space (i.e. it has an a index), and the non-Abelian transformation rotates the direction that it points in. In fact its kind of simple to understand what that term means: if you imagine the transformation field α^c as a vector pointing in $\mathbb{R}^3$, then that term as though the gauge field A_μ^a is trying to *rotate about that axis*, i.e.
$$\delta\vec{A}_\mu \approx \vec{A}_\mu \times \vec{\alpha} \tag{5.24}$$
3. As a brief aside: I will sometimes write A_μ with no superscript to mean $A_\mu \equiv A_\mu^a t^a$, i.e. it is a matrix-valued vector field.

5.3 The Yang-Mills field-strength and action

So we have understood how to take non-Abelian covariant derivatives, and we have seen that this requires us to introduce a non-Abelian gauge field A_μ^a. We would now like to build an action out of these gauge fields. We will do this in pretty much the same way as in the Abelian case: in particular, let's consider the commutator of two covariant derivatives acting on ψ. From the above discussion, we know that under a gauge transformation:

$$[D_\mu, D_\nu]\psi(x) \to U(x)[D_\mu, D_\nu]\psi(x) \equiv U(x)\left(-igF_{\mu\nu}^a t^a\right)\psi(x) \tag{5.25}$$

i.e. there are no pesky derivative terms. Thus this object has nice transformation properties and is a natural thing to think about. We will call it the field-strength tensor, and as the notation suggests, it is only a multiplicative operator acting on $\psi(x)$, not a differential one. Let's work it out:

$$[\partial_\mu - igA^a_\mu t^a, \partial_\nu - igA^b_\nu t^b]\psi = \left(-ig\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)t^a - g^2 A^a_\mu A^b_\nu [t^a, t^b]\right)\psi \tag{5.26}$$

$$= \left(-ig\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)t^a - ig^2 A^b_\mu A^c_\nu f^{bca} t^a\right)\psi \tag{5.27}$$

Comparing this to the definition, we see that in components the field-strength is

$$\boxed{F^a_{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) + gA^b_\mu A^c_\nu f^{abc}} \tag{5.28}$$

Importantly, from our previous study we know that under a gauge transformation we have

$$F_{\mu\nu} = F^{'a}_{\mu\nu} t^a = U(x) F^a_{\mu\nu} t^a U^\dagger(x)\ . \tag{5.29}$$

Thus $F_{\mu\nu}$ transforms in the adjoint. As explained above, the infinitesimal version of this transformation is

$$F^{'a}_{\mu\nu} = F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu} + \mathcal{O}(\alpha^2) \tag{5.30}$$

It is possible to check these properties explicitly by directly transforming the quantities in (5.28); there is a non-trivial cancellation between the terms that makes this possible, and thus the precise factors of g etc. are important.

However, now that we have this object, we can construct a non-Abelian-gauge-invariant action for the gauge field. This is essentially completely unique – just consider taking the trace of the matrix-valued field strength squared.

$$\boxed{S_{\rm YM}[A] = \int d^4x \left(-\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)\right)} \tag{5.31}$$

This is the celebrated Yang-Mills action. Let's check its properties: due to (5.29), under a gauge transformation we have

$$\mathrm{tr}(F'_{\mu\nu}F^{'\mu\nu}) = \mathrm{tr}\left(U(x)F_{\mu\nu}U^\dagger(x)U(x)F^{\mu\nu}U^\dagger(x)\right) = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{5.32}$$

and so it is clearly invariant. It's also convenient to work it out in components: we have

$$-\frac{1}{2}\mathrm{tr}\left(F^a_{\mu\nu}t^a F^{b\mu\nu}t^b\right) = -\frac{1}{2}F^a_{\mu\nu}F^{b\mu\nu}\mathrm{tr}\left(t^a t^b\right) \tag{5.33}$$

From your group theory course, it is always possible to pick the generators of $SU(N)$ so that we have

$$\mathrm{tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab} \tag{5.34}$$

which is a convention we will use – we then have that the action is

$$S_{\rm YM}[A] = \int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right) \tag{5.35}$$

where we sum over all the indices of the gauge field. This is it. Note that since $F^a_{\mu\nu}$ is non-linear in the field A_μ, this is a non-trivial interacting theory already. In the next section we will study the quantum theory – but first, let's study a few classical aspects.

Consider varying this action: we have

$$\delta S_{\rm YM}[A] = \int d^4x \left(-\frac{1}{2}F^{a\mu\nu}\delta F^a_{\mu\nu}\right) \tag{5.36}$$

$$= \int d^4x \left(-\frac{1}{2}F^{a\mu\nu}(\partial_\mu \delta A^a_\nu - \partial_\nu \delta A^a_\mu + g(\delta A^b_\mu A^c_\nu + A^b_\mu \delta A^c_\nu)f^{abc})\right) \tag{5.37}$$

$$= \int d^4x \left(\partial_\mu F^{a\mu\nu} + gf^{abc}A^b_\mu F^{c\mu\nu}\right)\delta A^a_\nu \tag{5.38}$$

From here we can read off that the equations of motion are

$$\partial_\mu F^{a\mu\nu} + g f^{abc} A_\mu^b F^{c\mu\nu} = 0 \tag{5.39}$$

Note that this equation is actually quite elegant – we are trying to take the derivative of an object $F_{\mu\nu}^a$ that transforms in the adjoint, so this is actually the gauge-covariant divergence $F_{\mu\nu}^a$:

$$\boxed{D_\mu F^{a\mu\nu} = 0} \tag{5.40}$$

This is the classical Yang-Mills equation. Despite the deceptive notation, it's a very nonlinear and messy equation.

We can of course also couple matter to these gauge fields. For example we started work with the fermions, so we can easily write down

$$S[A, \psi, \bar{\psi}] = \int d^4x \left(-\frac{1}{2}\text{tr}(F^2) + \bar{\psi}\left(i\not{D} - m\right)\psi \right) \tag{5.41}$$

where the gauge covariant derivative for the fermions is

$$D_\mu \psi = \partial_\mu \psi - igA_\mu^a t^a \psi \tag{5.42}$$

Note that again this is the most general renormalizable action, and it depends on only a single parameter g, the gauge coupling (which is hidden in the covariant derivatives).

5.4 Quantizing non-Abelian gauge theories

Thus motivated, we turn now to the quantum theory. Note that it *appears* that we have 3 types of massless gauge boson, one for each of the 3 generators of $SU(2)$. Let's now work out the Feynman rules for pure Yang-Mills theory, i.e.

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right) \tag{5.43}$$

We first recall that since we have

$$F_{\mu\nu}^a = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) + gA_\mu^b A_\nu^c f^{abc} \tag{5.44}$$

this theory is no longer quadratic, and thus even before we add any matter it is *already* interacting. Again, the interaction structure is completely fixed by gauge-invariance. Expanding out the action we see that we have

$$S[A] = \int d^4x \left(-\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gA_\mu^b A_\nu^c f^{abc}\right)\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gA_\mu^d A_\nu^e f^{ade}\right) \right) \tag{5.45}$$

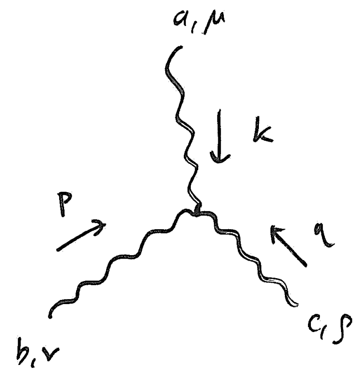
Thus there will be an interaction vertex that couples three A's and one that couples four of them.

We can work out the vertices through the usual techniques, which I will not go through in detail on the board as it is just bookkeeping. Note that this the first example we have seen of an interaction term that involves a derivative of the field, and this means that the Feynman rule will explicitly involve the momentum in the line connected to the vertex.

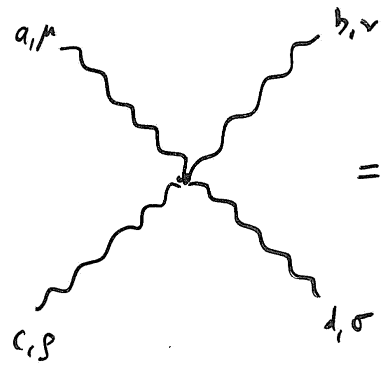
This structure is highly constrained – if you just wrote out the most general interaction there would be many other possible terms. In particular, note that it all depends on a single parameter g.

Now that we have all the vertices worked out, we need to again find a gauge-fixed propagator for the gauge field. After a great deal of work we would eventually find:

$$\boxed{D_{\mu\nu}^{ab}(p) = -\frac{i}{p^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\delta^{ab}} \tag{5.46}$$



$$= g f^{abc}\Big[\eta^{\mu\nu}(k-p)^{\rho} + \eta^{\nu\rho}(p-q)^{\mu} + \eta^{\rho\mu}(q-k)^{\nu}\Big]$$



$$= -ig^{2}\Big[f^{abe}f^{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right) + f^{ace}f^{bde}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right) + f^{ade}f^{bce}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}\right)\Big]$$

Figure 2: Interaction vertices for non-Abelian gauge fields

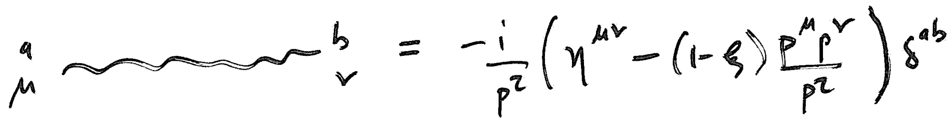


Figure 3: Propagator for non-Abelian gauge fields

exactly the same as the Abelian one, except with an extra δ^{ab} making it diagonal in gauge-field space. In this case the choice of gauge parameter $\xi = 1$ is called Feynman-t'Hooft gauge.

Here there is yet another wrinkle which I'm not telling you about: dealing with non-Abelian gauge theory requires keeping very careful track of the fact that only two of the four polarizations of A_μ are physical. The current way in which we deal with this is by introducing extra fields which "cancel out" the two polarizations that we don't want. These are called "ghosts". In fact, they were technically there in the Abelian theory as well, but I didn't talk about them because they don't interact with anything. In the non-Abelian theory they **do** and if you want to do a correct calculation involving fluctuations etc. you need to keep track of them.

A Group theory primer

Here is a little discussion of group theory to show how the ideas above work for any group. Consider an arbitrary group G. I denote an abstract set of generators of the group by T^a, where the index a runs over the Lie algebra of the group. Let me denote the dimension of the algebra by $d(G)$ (e.g. $d(SU(N)) = N^2 - 1$).

The T^a satisfy the following equation:

$$[T^a, T^b] = i f^{abc} T^c \tag{A.1}$$

where the f^{abc} are the *structure constants*.

Note that there is the Bianchi identity that all operations obey:

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \tag{A.2}$$

This identity together with the definition of the structure constants implies that they obey:

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \tag{A.3}$$

This is called the *Jacobi identity*.

Now there are many different representations for the group, i.e. the abstract T^a can be represented by concrete matrices. Let me denote the actual matrices in the fundamental representation of the group by little t^a. It is of course also true that

$$[t^a, t^b] = i f^{abc} t^c \tag{A.4}$$